

Hypercomplex structures on Kähler manifolds

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Abstract

Let (M, I) be a compact Kähler manifold admitting a hypercomplex structure (M, I, J, K) . We show that (M, I, J, K) admits a natural HKT-metric. This is used to construct a holomorphic symplectic form on (M, I) .

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1 Introduction

1.1 Hypercomplex manifolds

Let (M, I, J, K) be a manifold equipped with an action of the quaternion algebra \mathbb{H} on TM . The manifold M is called **hypercomplex** if the operators $I, J, K \in \mathbb{H}$ define integrable complex structures on M . As Obata proved ([Ob]), this condition is satisfied if and only if M admits a torsion-free connection ∇ preserving the quaternionic action:

$$\nabla I = \nabla J = \nabla K = 0.$$

Such a connection is called **an Obata connection on (M, I, J, K)** . It is necessarily unique ([Ob]).

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Hypercomplex manifolds were defined by C.P.Boyer ([Bo]), who gave a classification of compact hypercomplex manifolds for $\dim_{\mathbb{H}} M = 1$.

If the Obata connection ∇ , in addition, preserves a quaternionic Hermitian¹ metric g on M , then (M, I, J, K, g) is called **hyperkähler**. This definition is equivalent to the standard one, see e.g. [Bes].

It is unknown precisely which complex manifold admit hypercomplex structures.

Question 1.1: Consider a compact complex manifold (M, I) . Describe the set of hypercomplex structures (I, J, K) compatible with the given complex structure on M .

A similar question about hyperkähler structures is easily answered by the Calabi-Yau theorem. Recall that a hyperkähler manifold is holomorphically symplectic. Indeed, consider the 2-forms $\omega_J(\cdot, \cdot) = g(J\cdot, \cdot)$, $\omega_K(\cdot, \cdot) = g(K\cdot, \cdot)$; then

$$\Omega := \omega_J + \sqrt{-1} \omega_K \quad (1.1)$$

is a nowhere degenerate holomorphic $(2, 0)$ -form on (M, I) ([Bes]). A converse result is implied by Calabi-Yau theorem: a holomorphically symplectic compact Kähler manifold is necessarily hyperkähler.

Theorem 1.2: Let (M, I) be a compact holomorphically symplectic manifold with a Kähler form ω . Then there exists a unique hyperkähler metric g on M , with the same Kähler class as ω .

Proof: See [Bes]. ■

We have no similar description of complex manifolds admitting a hypercomplex structure. In this paper we study the following problem.

Question 1.3: Let (M, I) be a compact complex manifold of Kähler type². When (M, I) admits a hypercomplex structure?

¹A metric b is called **quaternionic Hermitian** if

$$g(Ix, Iy) = g(Jx, Jy) = g(Kx, Ky) = g(x, y)$$

for all $x, y \in TM$.

²That is, admitting a Kähler metric.

The following theorem gives an answer.

Theorem 1.4: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure. Then (M, I) is holomorphically symplectic.

Proof: In Subsection 1.2 we deduce Theorem 1.4 from Theorem 1.9, Theorem 1.10 and Theorem 1.11, which are proven in Sections 2, 3 and 4. ■

Remark 1.5: By Calabi-Yau theorem (Theorem 1.2), a holomorphically symplectic manifold admits a hyperkähler structure. However, the hypercomplex structure (M, I, J, K) on M can *a priori* have a different nature. The manifold (M, I, J, K) is hyperkähler if and only if the Obata connection ∇ preserves a metric. However, if the holonomy of ∇ is non-unitarian, such a metric does not exist.

Definition 1.6: Let (M, I) be a compact holomorphically symplectic Kähler manifold, and (M, I, J, K) a hypercomplex structure on (M, I) . Then (M, I, J, K) is called **exotic** if (M, I, J, K) is not hyperkähler, that is, if the holonomy of its Obata connection is not unitarian.

We conjecture that exotic hypercomplex structures do not exist.

1.2 HKT metrics and the canonical class

Let M be a hypercomplex manifold. A “hyperkähler with torsion” (HKT) metric on M is a special kind of a quaternionic Hermitian metric, which became increasingly important in mathematics and physics for the last 7 years.

HKT-metrics were introduced by P.S.Howe and G.Papadopoulos ([HP]) and much discussed in physics literature since then. For an excellent survey of these works written from a mathematician’s point of view, the reader is referred to the paper of G. Grantcharov and Y. S. Poon [GP].

The term “hyperkähler metric with torsion” is actually misleading, because an HKT-metric is not hyperkähler. This is why we prefer to use the abbreviation “HKT-manifold”.

Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian form, and Ω the $(2, 0)$ -form on (M, I) constructed from g as in (1.1). The hyperkähler condition can be written down as $d\Omega = 0$ ([Bes]). The HKT condition is weaker:

Definition 1.7: A quaternionic Hermitian metric is called an HKT-metric if

$$\partial(\Omega) = 0, \quad (1.2)$$

where $\partial : \Lambda_I^{2,0}(M) \longrightarrow \Lambda_I^{3,0}(M)$ is the Dolbeault differential on (M, I) , and Ω the $(2, 0)$ -form on (M, I) constructed from g as in (1.1).

It was shown in [HP], [GP], that this condition is in fact independent from the choice of the triple of complex structures (I, J, K) , $IJ = -JI = K$ in \mathbb{H} . In particular, we could replace the hypercomplex structure (M, I, J, K) with (M, J, K, I) . We obtain the following trivial claim

Claim 1.8: Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric. Consider g as a quaternionic Hermitian metric on a hypercomplex manifold (M, J, K, I) . Then g satisfies the HKT-condition on (M, I, J, K) if and only if g satisfies the HKT-condition on (M, J, K, I) . ■

HKT-metrics play in hypercomplex geometry the same role as the Kähler metrics play in complex geometry ([V1]).

The proof of Theorem 1.4 is split onto three steps, as follows.

Theorem 1.9: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure. Then there exists a finite non-ramified covering $\widetilde{M} \longrightarrow M$ such that the canonical bundle of (\widetilde{M}, I) is trivial as a holomorphic vector bundle.

Proof: See Section 2. ■

Theorem 1.10: Let (M, I, J, K) be a hypercomplex manifold. Assume that (M, I) admits a Kähler metric g . Then (M, I, J, K) admits an HKT-metric g_1 . Moreover, g_1 can be obtained by averaging g with the $SU(2)$ -action induced by quaternions.

Proof: See Section 3. ■

Theorem 1.11: Let (M, I, J, K) be a compact hypercomplex manifold admitting an HKT-metric. Assume that (M, I) admits a Kähler structure.

Assume, moreover, that there exists a finite non-ramified cover $\widetilde{M} \longrightarrow M$ such that the canonical bundle $K(\widetilde{M}, I)$ has a holomorphic trivialization. Then (M, I) is holomorphically symplectic.

Proof: See Section 4. ■

Theorem 1.11 concludes the proof of Theorem 1.4. Indeed, consider a compact hypercomplex manifold (M, I, J, K) , and assume that (M, I) admits a Kähler metric. By Theorem 1.9 the canonical class of (M, I) is trivial, by Theorem 1.10 (M, I, J, K) is HKT. We arrive at assumptions of Theorem 1.11, obtaining immediately that (M, I) is holomorphically symplectic.

2 Calabi-Yau theorem and triviality of canonical bundle

The following proposition is elementary.

Proposition 2.1: Let (M, I, J, K) , $\dim_{\mathbb{H}} M = n$ be a hypercomplex manifold, and

$$c_1(M, I) \in H^2(M, \mathbb{Z})$$

the first Chern class of (M, I) . Then $c_1(M, I) = 0$.

Proof: Let $SU(2) \subset \mathbb{H}^*$ be the group of unitary quaternions, acting on TM . A Riemannian metric g on M is quaternionic Hermitian if and only if g is $SU(2)$ -invariant. Taking an arbitrary Riemannian metric and averaging over $SU(2)$, we obtain a quaternionic Hermitian metric. We proved the following trivial claim

Claim 2.2: Let M be a hypercomplex manifold. Then M admits a quaternionic Hermitian metric. ■

Return to the proof of Proposition 2.1. To show that $c_1(M, I) = 0$, we need to construct a continuous trivialization of the canonical bundle $K(M, I) = \Lambda^{2n,0}(M, I)$, where $2n = \dim_{\mathbb{C}} M$. Let g be a quaternionic Hermitian metric on M , and

$$\Omega := g(J\cdot, \cdot) + \sqrt{-1} g(K\cdot, \cdot)$$

the corresponding non-degenerate $(2, 0)$ -form on (M, I) . Then

$$\Omega^n \in \Lambda_I^{2n,0}(M)$$

is a non-degenerate smooth section of the canonical bundle of $\Lambda_I^{2n,0}(M) = K(M, I)$ of (M, I) . Therefore, this bundle is topologically trivial. This gives $c_1(M, I) = 0$. ■

The classification of Kähler manifolds with vanishing c_1 ([Bo], [Bea], [Bes]) easily implies the following result.

Theorem 2.3: Let (M, I) be a compact Kähler manifold with

$$c_1(M, I) = 0.$$

Then there exists a finite non-ramified covering $\widetilde{M} \longrightarrow M$ such that the canonical bundle $K(\widetilde{M}, I)$ is trivial.

■

Combining Proposition 2.1 and Theorem 2.3, we obtain Theorem 1.9.

Remark 2.4: For a typical non-hyperkaehler compact hypercomplex manifold (M, I, J, K) , the complex manifold (M, I) admits no Kähler metrics, and the Calabi-Yau theorem cannot be applied. The canonical bundle $K(M, I)$ is trivial topologically by Proposition 2.1. However, it is in most cases non-trivial as a holomorphic vector bundle, even if one passes to a finite covering. It is possible to show that $K(M, I)$ is non-trivial for all hypercomplex manifold (M, I, J, K) such that (M, I) is a principal toric fibration over a base which has non-trivial canonical class; these include quasiregular Hopf manifolds and semisimple Lie groups with hypercomplex structure constructed by D. Joyce ([J]).

3 Kähler metrics and HKT metrics

Let (M, I, J, K) be hypercomplex manifold. Since J and I anticommute, J maps (p, q) -forms on (M, I) to (q, p) -forms:

$$J : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{q,p}(M).$$

Definition 3.1: Let $\eta \in \Lambda_I^{2,0}(M)$ be a $(2, 0)$ -form on (M, I) . Then η is called **J -real** if $J(\eta) = \overline{\eta}$, and **J -positive** if for any $x \in T^{1,0}(M, I)$,

$\eta(x, J(\bar{x})) \geq 0$. We say that η is **strictly J -positive** if this inequality is strict for all $x \neq 0$.

Denote the space of J -real, strictly J -positive $(2, 0)$ -forms by $\Lambda_{>0}^{2,0}(M, I)$.

We need the following linear-algebraic lemma, which is well known (see e.g. [V2]).

Lemma 3.2: Let M be a hypercomplex manifold. Then $\Lambda_{>0}^{2,0}(M, I)$ is in one-to-one correspondence with the set of quaternionic Hermitian metrics g on M . This correspondence is given by

$$g \longrightarrow g(J\cdot, \cdot) + \sqrt{-1} g(K\cdot, \cdot),$$

and the inverse correspondence by

$$\Omega \longrightarrow g(x, y) := \Omega(x, J(\bar{y})). \quad (3.1)$$

■

Lemma 3.3: Let (M, I, J, K) be a hypercomplex manifold, g_1 a Hermitian metric on (M, J) , $\omega_1 = g_1(\cdot, J\cdot)$ the corresponding differential 2-form, and Ω_1 the $\Lambda_I^{2,0}(M)$ -part of ω_1 . Then Ω_1 is strictly J -positive and J -real.

Proof: Since ω_1 is a $(1, 1)$ -form on (M, J) , we have $J(\omega_1) = \omega_1$. Therefore, $J(\Omega_1) = \bar{\Omega}_1$, and Ω_1 is J -real.

Given $x \in T_I^{1,0}(M)$, $x \neq 0$, the number $\Omega_1(x, J(\bar{x}))$ is real because Ω_1 is J -real. On the other hand,

$$\Omega_1(x, J(\bar{x})) = \omega_1(x, J(\bar{x})) = g_1(x, \bar{x}) > 0,$$

because Ω_1 is a $(2, 0)$ -part of ω_1 and $x, J(\bar{x})$ are $(1, 0)$ -vector fields. We have shown that Ω_1 is strictly J -positive. This proves Lemma 3.3. ■

We also have the following trivial claim

Claim 3.4: In assumptions of Lemma 3.3, let

$$\partial : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p+1,q}(M)$$

denote the standard Dolbeault differential ∂ on (M, I) . Then $\partial\Omega_1$ is the $(3, 0)$ -part of $d\omega_J$. In particular, if g_1 is Kähler on (M, J) then $\partial\Omega_1 = 0$.

Proof: By definition, Ω_1 is the $(2,0)$ -part of ω_J , and $\partial\Omega_1$ is the $(3,0)$ -part of $d\Omega_1$. ■

Remark 3.5: Let φ be a Kähler potential for the Kähler form ω_1 on (M, J) . By Claim 2.3 of [V2], on (M, I) we have $\Omega_1 = \partial\partial_J\varphi$, where $\partial_J = -J \circ \bar{\partial} \circ J$. The function φ satisfying $\Omega_1 = \partial\partial_J\varphi$ for an HKT-form Ω_1 is called an **HKT-potential** for an HKT-form Ω_1 .

Now, let (M, I, J, K) be a hypercomplex manifold, and g_1 a Kähler metric on (M, J) . Consider the form $\Omega_1 \in \Lambda_I^{2,0}(M)$ constructed above. Then Ω_1 is strictly J -positive and J -real by Lemma 3.3, and hence corresponds to a quaternionic Hermitian metric g on (M, I, J, K) . By Claim 3.4, $\partial\Omega_1 = 0$, hence g is HKT. Doing all calculations explicitly, a reader can show that g is obtained from g_1 by averaging over $SU(2)$ (we shall not use this claim). This proves Theorem 1.10. Indeed, in assumptions of Theorem 1.10 we are given a Kähler metric on (M, I) , so the above argument gives an HKT-metric on the hypercomplex manifold (M, J, K, I) ; this is equivalent to having an HKT-metric on (M, I, J, K) , as Claim 1.8 implies.

4 Supersymmetry on HKT-manifolds with trivial canonical class

Let (M, I, J, K, g) be an HKT-manifold, and $K(M, I)$ its canonical class. Using the quaternionic Hermitian metric g we trivialize the canonical class by a smooth non-degenerate section as in Proposition 2.1. Let $K^{1/2}$ be the square root of the canonical bundle corresponding to this trivialization. Writing $K(M, I)$ as a trivial bundle with the Chern connection $\nabla_{triv} + \theta$, we define $K^{1/2}$ as a trivial bundle with the connection $\nabla_{triv} + \frac{1}{2}\theta$. This connection is clearly induced by a holomorphic structure on $K^{1/2}$, and $K^{1/2} \otimes K^{1/2}$ is isomorphic to K as a holomorphic line bundle.

In [V1] we proved the following theorem, which is implied by an analogue of the Lefschetz-type $\mathfrak{sl}(2)$ -action in the HKT setting.

Theorem 4.1: Let (M, I, J, K) be a compact HKT-manifold, $\dim_{\mathbb{H}} M = n$, and $K^{1/2}$ the square root of a canonical bundle $K(M, I)$ constructed as above. Consider the Dolbeault class $[\bar{\Omega}] \in H_{\bar{\partial}}^{0,2}(M, I) = H^2(\mathcal{O}_{(M, I)})$ of $\bar{\Omega}$,

where $\Omega \in \Lambda_I^{2,0}(M)$ is the HKT-form of M , and let

$$H^l(K^{1/2}) \xrightarrow{\wedge [\bar{\Omega}]^{n-l}} H^{2n-l}(K^{1/2}) \quad (4.1)$$

be the corresponding multiplicative map on the holomorphic cohomology of $K^{1/2}$. Then (4.1) is an isomorphism.

Proof: In [V1] it was shown that the natural operator

$$L_\Omega : H^l(K^{1/2}) \longrightarrow H^{l+2}(K^{1/2})$$

belongs to an $\mathfrak{sl}(2)$ -triple. This is used in [V1] to obtain Theorem 4.1 in the same way as one obtains a similar result for the cohomology of a Kähler manifold. ■

When $K(M, I)$ is a trivial holomorphic bundle, $K^{1/2}$ is also a trivial bundle. We obtain that

$$\begin{aligned} &\text{when } K(M, I) \text{ is trivial, } [\Omega]^n \text{ is a generator of } H^{2n}(K^{1/2}) \cong \\ &H^0(K^{1/2})^* = \mathbb{C} \end{aligned} \quad (4.2)$$

(the last isomorphism is provided by the Serre's duality, using the triviality of the canonical bundle). Now we can prove Theorem 1.11.

Let (M, I, J, K) be a compact HKT-manifold, with \widetilde{M} a non-ramified finite covering of M with the canonical bundle $K(\widetilde{M}, I)$ trivial. Assume that (M, I) admits a Kähler metric. By Calabi-Yau theorem ([Yau]), (M, I) admits a Ricci-flat Kähler metric h . Let $\bar{\Omega}_h \in \Lambda_I^{0,2}(M)$ be a harmonic representative of the cohomology class $[\bar{\Omega}] \in H_{\bar{\partial}}^{0,2}(M, I)$ under h . Since (M, I) is Kähler, the harmonic $(2, 0)$ -form Ω_h is holomorphic. By Bochner-Lichnerowicz theorem ([Bes]), this implies

$$\nabla_h \Omega_h = 0, \quad (4.3)$$

where ∇_h is the Levi-Civita connection of h (this is true for any holomorphic form Ω_h on a Ricci-flat compact Kähler manifold). Let $\tilde{\Omega}_h, \tilde{\Omega}$ be Ω_h, Ω lifted to \widetilde{M} . By (4.2), $\tilde{\Omega}^n$, and hence $\tilde{\Omega}_h^n$, represents non-zero class in cohomology of (\widetilde{M}, I) . This implies $\Omega_h^n \neq 0$. By (4.3), we also have $\nabla_h \Omega_h^n = 0$, hence Ω_h^n trivializes $\Lambda_I^{2n,0}(M)$. We obtain that Ω_h is a non-degenerate holomorphic symplectic form on (M, I) . This proves Theorem 4.1. We finished the proof of Theorem 1.4.

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References

- [Bea] Beauville, A. *Varieties Kähleriennes dont la première classe de Chern est nulle*. J. Diff. Geom. **18**, pp. 755-782 (1983).
- [Bes] Besse, A., *Einstein Manifolds*, Springer-Verlag, New York (1987)
- [Bo] Bogomolov, F.A., *Hamiltonian Kähler manifolds*, Sov. Math., Dokl. **19**, 1462-1465 (1978).
- [Bo] Boyer, Charles P. *A note on hyper-Hermitian four-manifolds*. Proc. Amer. Math. Soc. 102 (1988), no. 1, 157-164.
- [GP] Grantcharov, G., Poon, Y. S., *Geometry of hyper-Kähler connections with torsion*, math.DG/9908015, also in Comm. Math. Phys. 213 (2000), no. 1, 19-37.
- [GH] Griffiths, Ph., Harris, J., *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
- [HP] Howe, P. S. Papadopoulos, G., *Twistor spaces for hyper-Kähler manifolds with torsion*, Phys. Lett. B 379 (1996), no. 1-4, 80-86.
- [J] Joyce, Dominic, *Compact hypercomplex and quaternionic manifolds*, J. Differential Geom. **35** (1992) no. 3, 743-761
- [Ob] Obata, M., *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Jap. J. Math., 26 (1955), 43-79.
- [V1] Verbitsky, M., *Hyperkähler manifolds with torsion, supersymmetry and Hodge theory*, math.AG/0112215, 47 pages, also in Asian J. of Math., Vol. 6 (4), pp. 679-712 (2002).
- [V2] Verbitsky, M., *Hyperkähler manifolds with torsion obtained from hyperholomorphic bundles*, Math. Res. Lett. 10 (2003), no. 4, 501-513, also in math.DG/0303129.
- [Yau] Yau, S. T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I.*, Comm. on Pure and Appl. Math. 31, 339-411 (1978).

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